# PROPERTIES OF CERTAIN HOMOGENEOUS LINEAR SUBSTITUTIONS.

By Harold Hilton.

1. Professor Loewy has discussed* the properties of a homogeneous linear substitution $A$

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \cdots + a_{tm}x_m \quad (t = 1, 2, \cdots, m),$$

which has as an invariant

$$\sum_t \epsilon_{tk}x_t\bar{x}_t \equiv x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_k\bar{x}_k - x_{k+1}\bar{x}_{k+1} - \cdots - x_m\bar{x}_m,$$

where $x$, $\bar{x}$ denote conjugate complex quantities, and $\epsilon_{ij} = 1$ when $i$, $j$ are both $> k$ or both $\leq k$, and $\epsilon_{ij} = -1$ when one of $i$ and $j$ is $> k$ and the other is $\leq k$ $(m \geq k \geq 0)$.

I shall call the substitution $A$ *quasi-unitary* in this case ("unitary" if $k = m$).

If $A$, instead of being quasi-unitary, satisfies the conditions $a_{ij} = \epsilon_{ij}\bar{a}_{ji}$ $A$ will be called *quasi-Hermitian* ("Hermitian" if $k = m$). For instance, the substitution with matrix

$$\begin{vmatrix} a & w & g & p \\ \bar{h} & b & f & q \\ \bar{g} & \bar{f} & c & r \\ -\bar{p} & -\bar{q} & -\bar{r} & s \end{vmatrix}$$

is quasi-Hermitian, if $a$, $b$, $c$, $s$ are real $(k = 3, m = 4)$.

The main interest of a quasi-Hermitian substitution lies in the fact that it bears a relation to a quasi-unitary substitution similar to that borne by a symmetric substitution to an orthogonal substitution. For instance, a quasi-Hermitian substitution is transformed by a quasi-unitary substitution into a quasi-Hermitian substitution, just as a symmetric substitution is transformed by an orthogonal substitution into a symmetric substitution;† and similar relations are developed in §§ 4 and 5.

But a quasi-Hermitian substitution has also properties analogous to important properties of a quasi-unitary substitution, as proved in §§ 2 and 3.

* Math. Annalen, 50 (1898), pp. 563, 564.

† Proc. London Math. Soc., 2, X (1911)., p. 274.

2. Professor Loewy (loc. cit.) has pointed out that those invariant-factors (elementartheiler) of a quasi-unitary substitution, which are not of the form $(\lambda - \alpha)^a$ where $\alpha\bar{\alpha} = 1$, can be grouped into pairs of the type $(\lambda - \alpha)^a$, $(\lambda - \bar{\alpha}^{-1})^a$ where $\alpha\bar{\alpha} \neq 1$.* We have similarly:—*Those invariant-factors of a quasi-Hermitian substitution which are not of the form* $(\lambda - \alpha)^a$ *where* $\alpha$ *is real, can be grouped into pairs of the type* $(\lambda - \alpha)^a$, $(\lambda - \bar{\alpha})^a$.

For suppose that $A$ is the quasi-Hermitian substitution

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \cdots + a_{tm}x_m \qquad (t = 1, 2, \cdots, m),$$

transformed by $P^{-1}$ into the canonical substitution $N$; so that $P^{-1}NP = A$. By a "canonical substitution" we mean a substitution of the type

$$x_t' = \lambda_t x_t + \beta_t x_{t+1} \qquad (t = 1, 2, \cdots, m),$$

where $\beta_t$ is 0 or 1, and is certainly 0 if $\lambda_t \neq \lambda_{t+1}$. It is well known that every substitution is transformable into such a canonical substitution.†

Suppose that $C$ is the Hermitian substitution

$$x_t' = c_{t1}x_1 + c_{t2}x_2 + \cdots + c_{tm}x_m \qquad (t = 1, 2, \cdots, m),$$

where

$$c_{ij} = \bar{c}_{ji} = \sum_{t=1}^{m} \epsilon_{tk} p_{ti} \bar{p}_{tj}.$$

Then we have (Proc. London Math. Soc., 2, X (1912), p. 282)

$$\lambda_i c_{ij} + \beta_{i-1} c_{i-1\, j} = \bar{\lambda}_j c_{ij} + \beta_{j-1} c_{i\, j-1}.$$

From this it follows, as in Mess. Math. (1912), p. 148, that, if $\lambda_i \neq \bar{\lambda}_j$, then $c_{ij} = 0$; but that if $\lambda_i = \bar{\lambda}_j$, then

$$\begin{aligned} c_{i-1\, j} &= 0 \quad \text{when} \quad \beta_{i-1} = 1 \quad \text{and} \quad \beta_{j-1} = 0, \\ c_{i\, j-1} &= 0 \quad \text{when} \quad \beta_{i-1} = 0 \quad \text{and} \quad \beta_{j-1} = 1, \\ c_{i-1\, j} &= c_{ij-1} \quad \text{when} \quad \beta_{i-1} = 1 \quad \text{and} \quad \beta_{j-1} = 1. \end{aligned}$$

For example, if $N$ is

$$x_1' = \alpha x_1 + x_2, \quad x_2' = \alpha x_2 + x_3, \quad x_3' = \alpha x_3; \quad x_4' = \alpha x_4 + x_5,$$
$$x_5' = \alpha x_5; \quad x_6' = \alpha x_6 + x_7, \quad x_7' = \alpha x_7; \quad x_8' = \alpha x_8,$$

where $\alpha$ is real, $C$ has a matrix of the type

* See also Proc. London Math. Soc., 2, XI (1912), p. 97.

† See, for instance, Mess. Math. (1909), p. 24.

$$\begin{vmatrix} 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & d & 0 & f & 0 \\ a & b & c & d & e & f & g & h \\ 0 & 0 & \bar{d} & 0 & i & 0 & k & 0 \\ 0 & \bar{d} & \bar{e} & i & j & k & l & m \\ 0 & 0 & \bar{f} & 0 & \bar{k} & 0 & n & 0 \\ 0 & \bar{f} & \bar{g} & \bar{k} & \bar{l} & n & p & q \\ 0 & 0 & \bar{h} & 0 & \bar{m} & 0 & \bar{q} & r \end{vmatrix},$$

$a, b, c, i, j, n, p, r$ being real;
or again if $N$ is

$$\left.\begin{aligned} x_1' &= \alpha x_1 + x_2, \quad x_2' = \alpha x_2; \quad x_3' = \alpha x_3; \\ x_4' &= \bar{\alpha} x_4 + x_5, \quad x_5' = \bar{\alpha} x_5; \quad x_6' = \bar{\alpha} x_6 \end{aligned}\right\}$$

$c$ has a matrix of the type

$$\begin{vmatrix} 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & a & b & d \\ 0 & 0 & 0 & 0 & f & e \\ 0 & \bar{a} & 0 & 0 & 0 & 0 \\ \bar{a} & \bar{b} & \bar{f} & 0 & 0 & 0 \\ 0 & \bar{d} & \bar{e} & 0 & 0 & 0 \end{vmatrix}.$$

From these examples the general form of $C$ is clear.

Now $\Sigma c_{ij} x_i \bar{x}_j$ is what $\sum_t \epsilon_{tk} x_t \bar{x}_t$ becomes when $A$ is transformed into $N$; and therefore the matrix of $C$ considered as a determinant does not vanish. But this is readily seen to be impossible unless the complex invariant-factors of $C$ are paired as stated in the above theorem.

3. The quasi-Hermitian substitution $A$ when transformed into canonical form $N$ becomes the direct product of substitutions of the form

$$\left.\begin{aligned} x_1' &= \alpha x_1 + x_2, \quad \cdots, \quad x_{s-1}' = \alpha x_{s-1} + x_s, \quad x_s' = \alpha x_s \\ y_1' &= \bar{\alpha} y_1 + y_2, \quad \cdots, \quad y_{s-1}' = \bar{\alpha} y_{s-1} + y_s, \quad y_s' = \bar{\alpha} y_s \end{aligned}\right\}$$

where $\alpha$ is not real, and of substitutions of the form

$$x_1' = \alpha x_1 + x_2, \quad \cdots, \quad x_{s-1}' = \alpha x_{s-1} + x_s, \quad x_s' = \alpha x_s$$

where $\alpha$ is real.

Then we can choose the new variables $x$, $y$, $X$ so that $A$ becomes $N$ and $\sum_t \epsilon_{tk} x_t \bar{x}_t$ becomes the sum of functions of the type

$$(x_1\bar{y}_s + \bar{x}_1 y_s) + (x_2\bar{y}_{s-1} + \bar{x}_2 y_{s-1}) + \cdots + (x_s\bar{y}_1 + \bar{x}_s y_1)$$

and

$$\pm (X_1\bar{X}_s + X_2\bar{X}_{s-1} + \cdots + X_{s-1}\bar{X}_2 + X_s\bar{X}_1)$$

respectively.

The proof of this statement is suppressed, since it is exactly similar to the proof of a similar theorem for symmetric substitutions published elsewhere.*

As in Proc. London Math. Soc., 2, XI (1911), pp. 98–100, we can show that when

$$(x_1\bar{y}_s + \bar{x}_1 y_s) + (x_2\bar{y}_{s-1} + \bar{x}_2 y_{s-1}) + \cdots + (x_s\bar{y}_1 + \bar{x}_s y_1)$$

is reduced by change of variables to the form

$$\pm z_1\bar{z}_1 \pm z_2\bar{z}_2 \pm z_3\bar{z}_3 \pm \cdots$$

there are $s$ positive and $s$ negative signs; while when

$$X_1\bar{X}_s + X_2\bar{X}_{s-1} + \cdots + X_s\bar{X}_1$$

is reduced to this form, the number of negative and the number of positive signs are equal when $s$ is even, and differ by unity when $s$ is odd. It follows that a series of properties of quasi-unitary substitutions established by Loewy (Math. Annalen, 50 (1898), pp. 563, 564), are also properties of a quasi-Hermitian substitution. For instance:—" The unreal characteristic-roots of the quasi-Hermitian substitution $A$ are not more than $2k'$ in number; $k'$ being the smaller of the quantities $k$ and $m-k$. If exactly $m-2k'$ characteristic-roots are real, they correspond to linear invariant-factors."

" $A$ cannot have more than $k'$ invariant-factors which are not linear. If $A$ has exactly $k'$ such invariant-factors they are all of degree 2 or 3. If they are all of degree 3, every characteristic-root of $A$ is real."

4. A relation between quasi-Hermitian and quasi-unitary substitutions similar to that between symmetric and orthogonal substitutions, and proved in the same way,† is the following:—

All quasi-Hermitian substitutions with given invariant-factors can be transformed by a quasi-unitary substitution into the same quasi-Hermitian substitution which is the direct product of substitutions each of which has a single invariant-factor $(\lambda - \alpha)^a$ where $\alpha$ is real, or a pair of invariant-

* Proc. London Math. Soc., 2, XII (1913), p. 94.

† Mess. Math. (1912), p. 146; Bôcher's Higher Algebra, p. 302.

factors $(\lambda - \alpha)^a$, $(\lambda - \bar{\alpha})^a$. Similarly for quasi-unitary substitutions transformed by quasi-unitary substitutions.

Example:—Find a quasi-Hermitian substitution with invariant-factors

$$(\lambda - \alpha)^2, \qquad (\lambda - \bar{\alpha})^2.$$

Since

$$x_1\bar{x}_4 + x_2\bar{x}_3 + x_3\bar{x}_2 + x_4\bar{x}_1 \equiv (x + \tfrac{1}{2}x_4)(\bar{x}_1 + \tfrac{1}{2}\bar{x}_4) + (x_2 + \tfrac{1}{2}x_3)(\bar{x}_2 + \tfrac{1}{2}\bar{x}_3)$$
$$- (x_2 - \tfrac{1}{2}x_3)(\bar{x}_2 - \tfrac{1}{2}\bar{x}_3) - (x_1 - \tfrac{1}{2}x_4)(\bar{x}_1 - \tfrac{1}{2}\bar{x}_4)$$

the required substitution is $F^{-1}NF$, where $N$ is the canonical substitution

$$x_1' = \alpha x_1 + x_2, \qquad x_2' = \alpha x_2; \qquad x_3' = \bar{\alpha}x_3 + x_4, \qquad x_4' = \bar{\alpha}x_4$$

and $F$ is

$$x_1' = x_1 + \tfrac{1}{2}x_4, \qquad x_2' = x_2 + \tfrac{1}{2}x_3, \qquad x_3' = x_2 - \tfrac{1}{2}x_3, \qquad x_4' = x_1 - \tfrac{1}{2}x_4.$$

A similar process will find a quasi-Hermitian or symmetric substitution with any assigned invariant-factors.

5. If we are given any substitution $A$, there are substitutions permutable with $A$ $k$-ply infinite in number, where $k$ is known when the invariant-factors of $A$ are given.* The problem suggests itself:—"To determine $k$ if the substitutions permutable with $A$ are limited in any way; if, for instance, they are orthogonal."

This problem can be solved in special cases. We have, for instance, the results:—

*If a symmetric substitution $A$ has $\alpha$ invariant-factors $(\lambda - \lambda_1)^a$, $\beta$ invariant-factors $(\lambda - \lambda_1)^b$, $\gamma$ invariant-factors $(\lambda - \lambda_1)^c$, $\cdots$, where*

$$a > b > c > \cdots,$$

*the orthogonal substitutions permutable with $A$ are*

$$\Sigma\tfrac{1}{2}\{\alpha(\alpha - 1)a + \beta(3\beta - 1)b + \gamma(5\gamma - 1)c + \delta(7\delta - 1)d + \cdots\}\text{-ply}$$

*infinite in number; the summation being extended over each distinct characteristic-root of $A$.*

*If a quasi-Hermitian substitution $A$ has $\rho$ invariant-factors $(\lambda - \lambda_1)^r$ and $(\lambda - \bar{\lambda}_1)^r$, $\sigma$ invariant-factors $(\lambda - \lambda_1)^s$ and $(\lambda - \bar{\lambda}_1)^s$, $\tau$ invariant-factors $(\lambda - \lambda_1)^t$ and $(\lambda - \bar{\lambda}_1)^t$, $\cdots$, $(r > s > t > \cdots)$, and $\alpha$ invariant-factors $(\lambda - \lambda_0)^a$, $\beta$ invariant factors $(\lambda - \lambda_0)^b$, $\gamma$ invariant-factors $(\lambda - \lambda_0)^c$, $\cdots$, $(a > b > c > \cdots)$, where $\lambda_0$ is real, the quasi-unitary substitutions permutable with $A$ are*

$$\{\Sigma[\rho^2 r + \sigma(2\rho + \sigma)s + \tau(2\rho + 2\sigma + \tau)t + \cdots]$$
$$+ \Sigma\tfrac{1}{2}[\alpha(\alpha - 1)a + \beta(3\beta - 1)b + \gamma(5\gamma - 1)c + \cdots]\}\text{-ply}$$

* Mess. Math. (1911), p. 112.

*infinite in number, the summations being extended over each pair of characteristic roots* $\lambda_1$ *and* $\overline{\lambda_1}$ *of* $A$ *and over each real characteristic-root* $\lambda_0$ *of* $A$.

I have obtained somewhat similar results for orthogonal substitutions permutable with a given orthogonal substitution, or quasi-unitary with quasi-unitary, in special cases; but the general result appears to be very complex.

The method of proof will be evident if we consider a particular example. Suppose, for instance, that the symmetric substitution $A$ has invariant-factors $(\lambda - \alpha)^s$, $(\lambda - \alpha)^s$, $(\lambda - \alpha)^s$. It is required to prove that the orthogonal substitutions permutable with $A$ are $\frac{1}{2}.3.2.s$-ply infinite in number.

Suppose that $A$, when transformed into canonical form $N = PAP^{-1}$, becomes

$$\left.\begin{array}{llll} x_1' = \alpha x_1 + x_2, & \cdots, & x_{s-1}' = \alpha x_{s-1} + x_s, & x_s' = \alpha x_s \\ \xi_1' = \alpha \xi_1 + \xi_2, & \cdots, & \xi_{s-1}' = \alpha \xi_{s-1} + \xi_s, & \xi_s' = \alpha \xi_s \\ X_1' = \alpha X_1 + X_2, & \cdots, & X_{s-1}' = \alpha X_{s-1} + X_s, & X_s' = \alpha X_s \end{array}\right\}.$$

Then if $P$ is chosen properly,

$$x_1^2 + x_2^2 + x_3^2 + \cdots$$

becomes

$$\varphi_1(x, x) + \varphi_1(\xi, \xi) + \varphi_1(X, X)^*$$

where

$$\left.\begin{array}{ll} \varphi_1(x, y) \equiv x_1 y_s + x_2 y_{s-1} + x_3 y_{s-2} + \cdots + x_{s-2} y_3 + x_{s-1} y_2 + x_s y_1 \\ \varphi_2(x, y) \equiv \qquad x_2 y_s + x_3 y_{s-1} + \cdots + x_{s-1} y_3 + x_s y_2 \\ \varphi_3(x, y) \equiv \qquad\qquad x_3 y_s + \cdots + x_s y_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi_s(x, y) \equiv \qquad\qquad\qquad x_s y_s \end{array}\right\}.$$

Since if $BA = AB$,

$$PBP^{-1} \cdot PAP^{-1} = PAP^{-1} \cdot PBP^{-1};$$

the number of substitutions permutable with $A$ is the same as the number permutable with $N$. It therefore suffices to prove that the substitutions permutable with $N$ and having $\varphi_1(x, x) + \varphi_1(\xi, \xi) + \varphi_1(X, X)$ as an invariant are $\frac{1}{2}.3.2.s$-ply infinite in number.

* Proc. London Math. Soc., 2, XII (1913), p. 94.

Now the most general substitutions permutable with $N$ will be

$$\left.\begin{aligned}
x_1' &= a_{11s}x_1 + a_{11\,s-1}x_2 + a_{11\,s-2}x_3 + \cdots + a_{111}x_s + a_{12s}\xi_1 + a_{12\,s-1}\xi_2 \\
&\quad + a_{12\,s-2}\xi_3 + \cdots + a_{121}\xi_s + a_{13s}X_1 + a_{13\,s-1}X_2 + a_{13\,s-2}X_3 \\
&\quad + \cdots + a_{131}X_s \\
x_2' &= a_{11s}x_2 + a_{11\,s-1}x_3 + \cdots + a_{112}x_s + a_{12s}\xi_2 + a_{12\,s-1}\xi_3 + \cdots \\
&\quad + a_{122}\xi_s + a_{13s}X_2 + a_{13\,s-1}X_3 + \cdots + a_{132}X_s \\
x_3' &= a_{11s}x_3 + \cdots + a_{113}X_s + a_{12s}\xi_3 + \cdots + a_{123}\xi_s + a_{13s}X_3 + \cdots + a_{133}X_s \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
x_s' &= a_{11s}x_s + a_{12s}\xi_s + a_{13s}X_s, \\
\xi_1' &= a_{21s}x_1 + \cdots + a_{211}x_s + a_{22s}\xi_1 + \cdots + a_{221}\xi_s + a_{23s}X_1 + \cdots + a_{231}X_s \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
X_1' &= a_{31s}x_1 + \cdots + a_{311}x_s + a_{32s}\xi_1 + \cdots + a_{321}\xi_s + a_{33s}X_1 + \cdots + a_{331}X_s \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}\right\}.$$

Operating with this on $\varphi_1(x, x) + \varphi_1(\xi, \xi) + \varphi_1(X, X)$ we get*

$$\begin{aligned}
\sum_t \{ & [\varphi_s(a_{t1}, a_{t1}) \cdot \varphi_1(x, x) + \varphi_s(a_{t2}, a_{t1}) \cdot \varphi_1(\xi, x) + \varphi_s(a_{t3}, a_{t1}) \cdot \varphi_1(X, x) \\
& + \varphi_s(a_{t1}, a_{t2})\varphi_1(x, \xi) + \varphi_s(a_{t2}, a_{t2}) \cdot \varphi_1(\xi, \xi) + \varphi_s(a_{t3}, a_{t2}) \cdot \varphi_1(X, \xi) \\
& + \varphi_s(a_{t1}, a_{t3}) \cdot \varphi_1(x, X) + \varphi_s(a_{t2}, a_{t3}) \cdot \varphi_1(\xi, X) + \varphi_s(a_{t3}, a_{t3}) \cdot \varphi_1(X, X)] \\
& + [\varphi_{s-1}(a_{t1}, a_{t1}) \cdot \varphi_2(x, x) + \cdots] + [\varphi_{s-2}(a_{t1}, a_{t1}) \cdot \varphi_3(x, x) + \cdots] + \cdots \}
\end{aligned}$$
$$(t = 1, 2, 3).$$

This must reduce to $\varphi_1(x, x) + \varphi_1(\xi, \xi) + \varphi_1(X, X)$.

Let $A_s$, $A_{s-1}$, $A_{s-2}$, $\cdots$ be the matrices whose general elements are respectively $a_{ij\,s}$, $a_{ij\,s-1}$, $a_{ij\,s-2}$, $\cdots$ $(i, j = 1, 2, 3)$.

Then we have in turn $A_sA_s' = 1$, $A_sA_{s-1}' + A_{s-1}A_s' = 0$, $A_sA_{s-2}' + A_{s-1}A_{s-1}' + A_{s-2}A_s' = 0$, $\cdots$.

Hence firstly, $A$ is orthogonal so that the quantities $a_{ijs}$ are functions of $\frac{1}{2}.3.2$ independent quantities.†

Then, when the quantities $a_{ijs}$ are fixed, the quantities $a_{ij\,s-1}$ are functions of $\frac{1}{2}.3.2$ independent quantities, since $A_sA_{s-1}' + A_{s-1}A_s' = 0$.

In fact, given any non-singular matrix $A$ of degree $m$, we can always find a matrix $B$ of degree $m$, such that $AB' + BA'$ is a given symmetric matrix $S$, in a $\frac{1}{2}m(m-1)$-ply infinite number of ways. For take $P$ any matrix whose elements $p_{ij}$ are arbitrary if $j > i$, and whose elements $p_{ij}$ are given by $S = P + P'$ when $i \leq j$. Then we may suppose $AB' = P$, $BA' = P'$; which gives $B$ in terms of the $\frac{1}{2}m(m-1)$ arbitrary quantities $p_{ij}$ $(j > i)$.

Then when the quantities $a_{ijs}$, $a_{ij\,s-1}$ are fixed, the quantities $a_{ij\,s-2}$ are functions of $\frac{1}{2}.3.2$ independent quantities, since $A_sA_{s-2}' + A_{s-1}A_{s-1}' + A_{s-2}A_s' = 0$. Continuing this process, we see that the required substitutions permutable with $N$ are $\frac{1}{2}.3.2.s$-ply infinite in number.

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* Proc. London Math. Soc., 2, XII (1913), p. 96

† Cayley, Crelle, XXXII (1846), p. 119, Collected Works, I, p. 332.